where

$$S_1(z,\chi) = \sum_{\rho<p\leq\rho'} \frac{\Lambda_{q^{\delta/4}}(p)\chi(p)}{p^s \log p} ,$$

$$S_2(z,\chi) = \sum_{\rho'<p\leq q} \frac{\Lambda_{q^{\delta/4}}(p)\chi(p)}{p^s \log p} ,$$

and $\rho' > \rho$. Clearly

$$\sum_{\chi\in\mathcal{L}_d(q)} |S_2(z,\chi)|^2 \leq \sum_{\chi \pmod q} |S_2(z,\chi)|^2$$

and the sum over all $\chi \pmod q$ may be estimated as in the case $K = 0$ above. We then obtain

(5) $$\sum_{\chi\in\mathcal{L}_d(q)} |S_2(z,\chi)|^2 << \phi(q)(\rho')^{1-2\sigma_1}$$

for $z \in \partial U$, where the implied constant depends on σ_1. We now estimate the remaining sum on the right hand side of (4). Fix $\varepsilon > 0$ so small that $0 < d-\varepsilon$ and $d+\varepsilon < \frac{1}{2}$. From the proof of Lemma 3.6 we recall the function $\xi(\tau)$ was defined so that $\xi(\tau) = 1$ if $\|\tau\| \leq d$, $\xi(\tau) = 0$ if $\|\tau\| \geq d+\varepsilon$, and $\xi(\tau)$ drops off to 0 linearly on $d \leq \|\tau\| \leq d+\varepsilon$. Also

(6) $$\xi(\tau) = \sum_{n=-\infty}^{\infty} c_n e(\tau n) ,$$

where the series converges absolutely and

$$c_0 = 2d+\varepsilon \ .$$

Let $p_1,\ldots,p_K$ denote the primes in $\mathcal{P}_K$ and define

$$\xi_K(\chi) = \prod_{k=1}^{K} \xi\left(\frac{\arg \chi(p_k)}{2\pi} - \theta_{p_k}\right) .$$

Then

$$(7) \qquad 0 \leq \xi_K^2(\chi) \leq \xi_K(\chi) \qquad (\chi \pmod q)$$

and

$$(8) \qquad 1 = \xi_K^2(\chi) = \xi_K(\chi) \qquad (\chi \in \mathcal{L}_d(q)) \ .$$

Furthermore we have

$$(9) \quad \xi_K(\chi) = \sum_{-\infty<n_1,\ldots,n_K<\infty} c_{n_1}\cdots c_{n_K} e\left(-\sum_{k=1}^{K} n_k\theta_{p_k}\right)\chi\left(\prod_{k=1}^{K} p_k^{n_k}\right)$$

$$= \sum_{\vec{n}} c(\vec{n})\chi\left(\prod_{k=1}^{K} p_k^{n_k}\right) \quad , \text{ say,}$$

where $\vec{n} = (n_1,\ldots,n_K)$ runs through all K-tuples of integers and

$$(10) \qquad c(\vec{0}) = c_0^K = (2d+\varepsilon)^K \ .$$

Since the series in (6) is absolutely convergent, so is the series in (9). Thus, given $\varepsilon_1 > 0$, there exists a number $N(\varepsilon_1)$ such that for each $N \geq N(\varepsilon_1)$ and all $\chi \pmod q$,

$$\left|\xi_K(\chi) - \sum_{\|\vec{n}\| \leq N} c(\vec{n})\chi\left(\prod_{k=1}^{K} p_k^{n_k}\right)\right| < \varepsilon_1 ,$$

where $\|\vec{n}\| = \left(\prod_{k=1}^{K} n_k^2\right)^{1/2}$. For such an N we obtain

$$\xi_K^2(\chi) \leq 2\left|\sum_{\|\vec{n}\| \leq N} c(\vec{n})\chi\left(\prod_{k=1}^{K} p_k^{n_k}\right)\right|^2$$

$$+ 2\left|\xi_K(\chi) - \sum_{\|\vec{n}\| \leq N} c(\vec{n})\chi\left(\prod_{k=1}^{K} p_k^{n_k}\right)\right|^2$$

$$\leq 2\left|\sum_{\|\vec{n}\| \leq N} c(\vec{n})\chi\left(\prod_{k=1}^{K} p_k^{n_k}\right)\right|^2 + 2\varepsilon_1^2 .$$

Using this, (7), and (8), we find

$$\text{(11)} \quad \sum_{\chi \in \mathcal{L}_d(q)} |S_1(z,\chi)|^2 \leq \sum_{\chi \pmod q} \xi_K^2(\chi)|S_1(z,\chi)|^2$$

$$< 2 \sum_{\chi \pmod q} \left|\sum_{\|\vec{n}\| \leq N} c(\vec{n})\chi\left(\prod_{k=1}^{K} p_k^{n_k}\right)\right|^2 |S_1(z,\chi)|^2$$

$$+ 2\varepsilon_1^2 \sum_{\chi \pmod q} |S_1(z,\chi)|^2 .$$

Write $z = x+iy$. We immediately obtain from Lemma 5.2 that

$$(12) \qquad \sum_{\chi \pmod q} |S_1(z,\chi)|^2 = \phi(q) \sum_{\rho<p\leq\rho'} \frac{|\Lambda_{q^{\delta/4}}(p)\chi(p)/\log p|^2}{p^{2x}} .$$

As in our previous estimates we see that this leads to

$$(13) \qquad \sum_{\chi \pmod q} |S_1(z,\chi)|^2 << \phi(q)\rho^{1-2\sigma_1}$$

for $z \in \partial U$, where the constant depends on σ_1 . Again by Lemma 5.2 we have

$$(14) \qquad \sum_{\chi \pmod q} \Big| \sum_{||\vec{n}|| \leq N} c(\vec{n})\chi\Big(\prod_{k=1}^{K} p_k^{n_k}\Big)\Big|^2 |S_1(z,\chi)|^2$$

$$= \phi(q) \sum_{||\vec{n}|| \leq N} |c(\vec{n})|^2 \sum_{\rho<p\leq\rho'} \frac{|\Lambda_{q^{\delta/4}}(p)\chi(p)/\log p|^2}{p^{2x}} ,$$

provided q is so large that none of the numbers $p\prod_{k=1}^{K} p_k^{n_k}$, $\rho < p < \rho'$, are larger than q . The sum over p is exactly the right-hand side of (12), so by (13) it is

$$<< \phi(q)\rho^{1-2\sigma_1}$$

for $z \in \partial U$, where the constant depends on σ_1 . We deduce from (7) and (10) the absolute convergence of the series for

$\xi_K(\chi)$, and the relations

$$\lim_{q\to\infty} \frac{1}{\phi(q)} \sum_{\chi(\text{mod } q)} \chi(n)\overline{\chi}(m) = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{otherwise} \end{cases} ,$$

$$\lim_{q\to\infty} \frac{1}{\phi(q)} \sum_{\chi(\text{mod } q)} \chi(n) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{otherwise} \end{cases} ,$$

that

$$\sum_{\|\vec{n}\|\le N} |c(\vec{n})|^2 \le \sum_{\|\vec{n}\|} |c(\vec{n})|^2 = \lim_{q\to\infty} \frac{1}{\phi(q)} \sum_{\chi(\text{mod } q)} \xi_K^2(\chi)$$

$$\le \lim_{q\to\infty} \frac{1}{\phi(q)} \sum_{\chi(\text{mod } q)} \xi_K(\chi)$$

$$= c(\vec{0}) = (2d+\varepsilon)^K .$$

Thus the expressions in (14) are

$$<< \quad (2d+\varepsilon)^K \phi(q) \rho^{1-2\sigma_1} .$$

Combining this with (11) and (13) we get

$$(15) \qquad \sum_{\chi \varepsilon \mathcal{X}_d(q)} |S_1(z,\chi)|^2 << ((2d+\varepsilon)^K + \varepsilon_1^2) \phi(q) \rho^{1-2\sigma_1}$$

for each $z \in \partial U$, where the constant depends on σ_1. Finally, from (3), (4), (5), and (15) we find that

$$\sum_{\chi \in \mathcal{J}_d(q)} \max_{s \in C} |S(s,\chi)|^2$$

$$\ll ((2d+\varepsilon)^K + \varepsilon_1^2)\phi(q)\rho^{1-2\sigma_1} + \phi(q)(\rho')^{1-2\sigma_1} ,$$

where the constant depends on σ_1, δ, and A, or, since δ and A depend on σ_1, σ_2, and C, on σ_1, σ_2, and C. Choose ρ' so large that

$$(\rho')^{1-2\sigma_1} \leq (2d)^K \rho^{1-2\sigma_1} ,$$

and $\varepsilon, \varepsilon_1$ so small that

$$(2d+\varepsilon)^K + \varepsilon_1^2 \leq 2(2d)^K .$$

This gives (2) and proves the lemma. $\square$

Lemma 5.5. Let C be a compact set in the strip $\frac{1}{2} < \sigma_1 < \sigma < \sigma_2 < 1$, let $\rho \geq 1$, and let $0 < d < \frac{1}{2}$. For each $p \leq \rho$ let θ_p be fixed, $0 \leq \theta_p < 1$. If for $q \geq 1$, $\mathcal{J}(q)$ is a subset of the set of characters (mod q) with cardinality

$$J(q) = \phi(q)(1 + o(1)) ,$$

then for q sufficiently large there is a $\chi \in \mathcal{J}(q)$ such that

$$(16) \qquad \left\|\frac{\arg \chi(p)}{2\pi} - \theta_p\right\| \leq d \ , \ p \leq \rho \ , \ p \nmid q$$

and

$$\max_{s \in C}\left|\sum_{\rho < p \leq q} \frac{\Lambda_{q^{\delta/4}}(p)\chi(p)}{p^s \log p}\right| \ll \rho^{1/2-\sigma_1} .$$

The implicit constant depends on σ_1 , σ_2 , and C .

Proof: Let $K(q)$ be the number of primes $p \leq \rho$ such that $p \nmid q$, and let $\mathcal{Q}_d(q)$ be the set of characters (mod q) which satisfy (16). We write

$$\mathcal{H}_d(q) = \{\chi \in \mathcal{Q}_d(q) \,|\, \max_{s \in C}|S(s,\chi)| \leq c\rho^{1/2-\sigma_1}\} ,$$

where

$$S(s,\chi) = \sum_{\rho < p \leq q} \frac{\Lambda_{q^{\delta/4}}(p)\chi(p)}{p^s \log p} ,$$

and we write $G_d(q)$ for the cardinality of $\mathcal{H}_d(q)$. It suffices to show that for some choice of c depending on σ_1 , σ_2 , and C , and for all large q ,

$$\mathcal{H}_d(q) \cap \mathcal{J}(q)$$

is non-empty. Now for large q we have by Lemma 5.4 that

$$\sum_{\chi \in \mathcal{L}_d(q)} \max_{s \in C} |S(s,\chi)|^2 \ll (2d)^{K(q)} \phi(q) \rho^{1-2\sigma_1} ,$$

where the constant implied by $\ll$ depends on σ_1 , σ_2 , and C . It follows that the cardinality of the complement of $\mathcal{G}_d(q)$ in $\mathcal{L}_d(q)$ is

$$\ll \frac{(2d)^{K(q)}}{c^2} \phi(q) .$$

Thus, choosing c large enough (depending on σ_1 , σ_2 , and C) we can ensure that

$$G_d(q) > I_d(q) - \tfrac{1}{4}(2d)^{K(q)} \phi(q) ,$$

where $I_d(q)$ is the cardinality of $\mathcal{L}_d(q)$. By Lemma 5.3,

$$I_d(q) > \tfrac{1}{2}(2d)^{K(q)} \phi(q)$$

if q is large enough. Therefore

$$G_d(q) > \tfrac{1}{4}(2d)^{K(q)} \phi(q) .$$

Since

$$J(q) = \phi(q)(1 + o(1)) ,$$

$\mathcal{J}(q)$ and $\mathcal{H}_d(q)$ must overlap for large q. This proves the lemma. □

It is now necessary to define $\log L(s,\chi)$. Assume that $\chi \pmod q$ is not the principal character. For each zero $\beta+i\gamma$ of $L(s,\chi)$ with $\beta > \frac{1}{2}$, we remove the segment $(\frac{1}{2}+i\gamma, \beta+i\gamma]$ from the half-plane $\sigma > \frac{1}{2}$. We define $\log L(s,\chi)$ on the resulting slit half-plane to be that analytic branch of logarithm which satisfies

$$\lim_{\sigma\to\infty} \log L(s,\chi) = 0 .$$

With this convention we state

Lemma 5.6. Let C be a compact set in the strip $\frac{1}{2} < \sigma_1 < \sigma < \sigma_2 < 1$ and set $\delta = \sigma_1 - \frac{1}{2}$. For each $q \geq 1$ there is a subset $\mathcal{J}(q)$ of the set of characters (mod q) with cardinality

$$J(q) = \phi(q)(1 + o(1))$$

such that when q is sufficiently large,

$$\text{(17)} \quad \max_{s\in C} \Big| \log L(s,\chi) - \sum_{n<q^{\delta/2}} \frac{\Lambda_{q^{\delta/4}}(n)\chi(n)}{n^s \log n} \Big| \ll q^{-\delta^2/8}$$

for all $\chi \in \mathcal{J}(q)$. The implied constant depends on σ_1 and C.

Proof: Let $A = \max_{s \in C} |s|$. We define $\mathcal{J}(q)$ to be the set of characters remaining after we remove from the set of all characters (mod q) the principal character and any character whose corresponding L-function has a zero with

$$\beta \geq \frac{1}{2} + \frac{\delta}{2} , \ |\gamma| \leq A + q^{\delta/4} . \tag{18}$$

In this way we remove

$$<< \sum_{\chi (\mathrm{mod}\ q)} N(\tfrac{1}{2}+\tfrac{\delta}{2}, \ A+q^{\delta/4}, \ \chi)$$

characters. By Lemma 3.1 this is

$$<< \ (q(A + q^{\delta/4}))^{1-\delta/2} \ \log^{14}(q(A+q^{\delta/4}))$$

$$<< \ q^{1-\delta/4} \ .$$

Thus the cardinality of $\mathcal{J}(q)$ is

$$J(q) = \phi(q) + O(q^{1-\delta/4}) \ .$$

In view of Lemma 5.1 we therefore have

$$J(q) = \phi(q)(1 + o(1)) \ .$$

We note that for each $\chi \in \mathcal{J}(q)$, $L(s,\chi)$ has no zeros or poles in the region

$$\frac{1}{2}+\frac{\delta}{2} \le \sigma < \infty \ , \ |t| \le A+q^{\delta/4} \ .$$

Thus $\log L(s,\chi)$ is analytic and single-valued in this region.

We now prove (17), Let $\chi \in \mathcal{X}(q)$ and $s = \sigma+it \in C$. Integrating both sides of (3.19) in Lemma 3.8 along the half-line $[\sigma+it, \infty+it)$ yields

$$\log L(s,\chi) - \sum_{n<x^2} \frac{\Lambda_x(n)\chi(n)}{n^s \log n}$$

$$<< \frac{1}{\log x}\int_\sigma^\infty \sum_{r=0}^{\infty} \frac{x^{-2r-\alpha-u}+x^{-2(2r+\alpha+u)}}{(2r+\alpha+u)^2+t^2}\,du$$

$$+ \frac{1}{\log x}\int_\sigma^\infty \sum_{\beta+i\gamma} \frac{x^{\beta-u}+x^{2(\beta-u)}}{(\beta-u)^2+(\gamma-t)^2}\,du \ .$$

We write this as

$$\text{(19)} \quad \log L(s,\chi) - \sum_{n<x^2} \frac{\Lambda_x(n)\chi(n)}{n^s \log n} << \frac{1}{\log n}(R_1(s,\chi)+R_2(s,\chi)) \ .$$

We now estimate R_1 and R_2 assuming that $x \ge 2$. For R_1 we have

$$R_1(s,\chi) \ll \int_\sigma^\infty \left(\sum_{r=0}^\infty x^{-2r-\alpha-u}\right) du$$

$$\ll \left(\sum_{r=0}^\infty x^{-2r}\right)\int_\sigma^\infty x^{-u} du$$

$$\ll \frac{x^{-\sigma}}{\log x} ,$$

where the $\ll$ is absolute since $x \geq 2$. Thus, since $\sigma > \frac{1}{2}+\delta$ for $s \in C$, we find

(20)
$$R_1(s,\chi) \ll \frac{x^{-1/2-\delta}}{\log x}$$

uniformly for $s \in C$ and $\chi \in \mathcal{X}(q)$. To treat R_2 we write

(21)
$$R_2(s,\chi) = R_{21}(s,\chi) + R_{22}(s,\chi) ,$$

where

$$R_{21}(s,\chi) = \int_\sigma^\infty \sum_{|\gamma-t| \leq q^{\delta/4}} \frac{x^{\beta-u} + x^{2(\beta-u)}}{(\beta-u)^2 + (\gamma-t)^2} du$$

and

$$R_{22}(s,\chi) = \int_\sigma^\infty \sum_{q^{\delta/4}<|\gamma-t|} \frac{x^{\beta-u} + x^{2(\beta-u)}}{(\beta-u)^2 + (\gamma-t)^2} du .$$

In light of (18), each zero $\beta+i\gamma$ in the sum for $R_{21}(s,\chi)$ has $\beta < \frac{1}{2}+\frac{\delta}{2}$. Also for $u \geq \sigma$, $\beta-u < -\frac{\delta}{2}$. Thus

$$R_{21}(s,\chi) \ll \int_{\sigma}^{\infty} x^{\beta-u}\, du \sum_{|\gamma-t|\leq q^{\delta/4}} \frac{1}{\delta^2 + (\gamma-t)^2}$$

$$\ll \frac{x^{-\delta/2}}{\log x}\left(\sum_{|\gamma-t|\leq 1} \frac{1}{\delta^2} + \sum_{1<|\gamma-t|\leq q^{\delta/4}} \frac{1}{(\gamma-t)^2} \right) .$$

By Lemma 3.2,

$$\sum_{|\gamma-t|\leq 1} \frac{1}{\delta^2} \ll \frac{1}{\delta^2} \log q(|t|+2) \ll \log q(|t|+2)$$

and

$$\sum_{1<|\gamma-t|\leq q^{\delta/4}} \frac{1}{(\gamma-t)^2} \ll \sum_{j=1}^{[q^{\delta/4}]} \frac{\log q(|t+j|+2)}{j^2}$$

$$+ \sum_{j=1}^{[q^{\delta/4}]} \frac{\log q(|t-j|+2)}{j^2}$$

$$\ll \log q(|t|+q^{\delta/4}+2) .$$

Hence

$$R_{21}(s,\chi) \ll \frac{x^{-\delta/2}}{\log x} \log q(|t|+q^{\delta/4}+2) . \tag{22}$$

The implied constant depends only on δ . Since $\beta < 1$ for every zero $\beta+i\gamma$, we find that

$$R_{22}(s,\chi) << \int_{\sigma}^{\infty} (x^{1-u} + x^{2(1-u)})\,du \sum_{q^{\delta/4}<|\gamma-t|} \frac{1}{(\gamma-t)^2} .$$

The integral is

$$<< \frac{x^{1-2\delta}}{\log x}$$

as $\sigma > \sigma_1 = \frac{1}{2}+\delta$ and the constant is absolute. For the sum we obtain by Lemma 3.2 the estimate

$$\sum_{q^{\delta/4}<|\gamma-t|} \frac{1}{(\gamma-t)^2} << \sum_{j>[q^{\delta/4}]} \frac{\log q(|t+j|+2)}{j^2}$$

$$+ \sum_{j>[q^{\delta/4}]} \frac{\log q(|t-j|+2)}{j^2}$$

$$<< \sum_{j>[q^{\delta/4}]} \frac{\log q(|t|+j+2)}{j^2}$$

$$<< \int_{q^{\delta/4}}^{\infty} \frac{\log q(|t|+x+2)}{x^2}\,dx$$

$$= - \frac{\log q(|t|+x+2)}{x}\Big|_{q^{\delta/4}}^{\infty} + \int_{q^{\delta/4}}^{\infty} \frac{dx}{x(|t|+x+2)}$$

$$\ll \frac{\log q(|t|+q^{\delta/4}+2)}{q^{\delta/4}} \quad .$$

Thus

$$R_{22}(s,\chi) \ll \frac{x^{1-2\delta}}{\log x} \frac{\log q(|t|+q^{\delta/4}+2)}{q^{\delta/4}} \quad ,$$

where the constant is absolute. Combining this with (21) and (22) yields

$$R_2(s,\chi) \ll \frac{\log q(|t|+q^{\delta/4}+2)}{\log x} (x^{-\delta/2} + \frac{x^{1-2\delta}}{q^{\delta/4}}) \quad .$$

This along with (19) and (20) leads to

$$(23) \quad \log L(s,\chi) - \sum_{n<x^2} \frac{\Lambda_x(n)\chi(n)}{n^s \log n}$$

$$\ll \frac{1}{(\log x)^2}(x^{-1/2-\delta} + (x^{-\delta/2} + \frac{x^{1-2\delta}}{q^{\delta/4}})\log q(|t|+q^{\delta/4}+2)) \quad ,$$

where $x \geq 2$ and the implied constant depends only on δ, hence on σ_1. Taking $x = q^{\delta/4}$, $q \geq 2^{4/\delta}$, and noting that $|t| \leq A$ for $s \in C$, we find the right-hand side of (23) is

$$\ll \frac{1}{\delta^2(\log q)^2}\left(q^{-\delta/8-\delta^2/4} + (q^{-\delta^2/8} + q^{-\delta^2/2})\log q(A+q^{\delta/4}+2)\right)$$

$$\ll q^{-\delta^2/8} \quad .$$

This last $\ll$ depends on σ_1 and A, or σ_1 and C. Since this bound is uniform for $s \in C$ and $\chi \in \mathcal{J}(q)$, this establishes (17) and completes the proof of the lemma. □

Lemma 5.7. Let C be a compact set in the strip $\frac{1}{2} < \sigma_1 < \sigma < \sigma_2 < 1$ and set $\delta = \sigma_1 - \frac{1}{2}$. Suppose that $\rho > \mu > 1$. Then there exist entire functions $\ell_\mu(s,q)$ such that if $0 \le \theta_p < 1$ for $\mu < p \le \rho$, and if q is sufficiently large, there is a character $\chi \pmod q$ for which we have

$$\max_{s\in C} \Big|\log L(s,\chi) - \ell_\mu(s,q) - \sum_{\substack{\mu<p\le\rho \\ p\nmid q}} \frac{e(\theta_p)}{p^s}\Big| \ll \mu^{-\delta} .$$

The implicit constant depends on σ_1, σ_2, and C.

<u>Proof</u>: Let $N = [8 \log \mu]$ and take q so large that $q^{\delta/4} \ge \max(\mu^N, \rho)$. With $\Lambda_x(n)$ as in Lemma 5.6 we have

$$(24) \quad \sum_{n<q^{\delta/2}} \frac{\Lambda_{q^{\delta/4}}(n)\chi(n)}{n^s \log n} = \sum_{p\le\mu} \sum_{k=1}^{N} \frac{\chi(p^k)}{k\, p^{ks}} + \sum_{\mu<p\le\rho} \frac{\chi(p)}{p^s}$$

$$+ \sum_{\rho<p\le q^{\delta/2}} \frac{\Lambda_{q^{\delta/4}}(p)\chi(p)}{p^s \log p}$$

$$+ \sum_{p \le \mu} \sum_{k=N+1}^{\infty} \frac{\Lambda_{q^{\delta/4}}(p^k)\chi(p^k)}{p^{ks} \log p^k}$$

$$+ \sum_{\mu < p \le q^{\delta/2}} \sum_{k=2}^{\infty} \frac{\Lambda_{q^{\delta/4}}(p^k)\chi(p^k)}{p^{ks} \log p^k} .$$

Assume $s \in C$ and note that $|\Lambda_{q^{\delta/4}}(n)\chi(n)/\log n| \le 1$.

Then

$$\sum_{p \le \mu} \sum_{k=N+1}^{\infty} \frac{\Lambda_{q^{\delta/4}}(p^k)\chi(p^k)}{p^{ks} \log p^k} << \sum_{p \le \mu} \sum_{k=N+1}^{\infty} \frac{1}{p^{\sigma k}}$$

$$<< \sum_{p \le \mu} \frac{1}{p^{\sigma(N+1)}} << \mu 2^{-\sigma(N+1)} .$$

Since $\sigma > \frac{1}{2}$ for $s \in C$, $N = [8 \log \mu]$, and $\log 2 > \frac{1}{2}$, this is

$$<< \mu e^{-4 \log 2 \log \mu} = \mu^{1-4 \log 2} < \mu^{-1} .$$

Notice that the implicit constants in these estimates are absolute. Since $\sigma > \sigma_1 = \frac{1}{2}+\delta$ for $s \in C$, we next find

$$\sum_{\mu<p\le q^{\delta/2}} \sum_{k=2}^{\infty} \frac{\Lambda_{q^{\delta/4}}(p^k)\chi(p^k)}{p^{ks}\log p^k} << \sum_{\mu<p\le q^{\delta/2}} \sum_{k=2}^{\infty} \frac{1}{p^{\sigma_1 k}}$$

$$<< \sum_{\mu<p} \frac{1}{p^{2\sigma_1}} < \sum_{\mu<n} \frac{1}{n^{2\sigma_1}} << \frac{\mu^{1-2\sigma_1}}{2\sigma_1-1}$$

$$<< \mu^{-2\delta} .$$

The implicit constant in the last $<<$ depends on σ_1. As

$$\mu^{-1} < \mu^{-2\delta} ,$$

we may combine these estimates with (24) to obtain

$$(25) \quad \sum_{n<q^{\delta/2}} \frac{\Lambda_{q^{\delta/4}}(n)\chi(n)}{n^s \log n} - \sum_{p\le\mu} \sum_{k=1}^{N} \frac{\chi(p)^k}{k\, p^{ks}} - \sum_{\mu<p\le\rho} \frac{\chi(p)}{p^s}$$

$$= \sum_{\rho<p\le q^{\delta/2}} \frac{\Lambda_{q^{\delta/4}}(p)\chi(p)}{p^s \log p} + O(\mu^{-2\delta})$$

uniformly for $s \in C$ and $\chi \pmod q$, where the constant in the O-term depends on σ_1. Now let $\mathcal{X}(q)$ be the set of characters in Lemma 5.6. By that lemma when q is sufficiently large, we have

$$(26) \quad \sum_{n<q^{\delta/2}} \frac{\Lambda_{q^{\delta/4}}(n)\chi(n)}{n^s \log n} = \log L(s,\chi) + O(q^{-\delta^2/8})$$

uniformly for $s \in C$ and $\chi \in \mathcal{J}(q)$, where the O-term constant depends on σ_1 and C. Furthermore since the cardinality of $\mathcal{J}(q)$ is $\phi(q)(1 + o(1))$, Lemma 5.5 implies that if $0 < d < \frac{1}{2}$ and q is sufficiently large, then there is a $\chi \in \mathcal{J}(q)$ such that

$$(27) \qquad \left\| \frac{\arg \chi(p)}{2\pi} \right\| \leq d \quad \text{if} \quad p \leq \mu \quad \text{and} \quad p \nmid q ,$$

$$(28) \qquad \left\| \frac{\arg \chi(p)}{2\pi} - \theta_p \right\| \leq d \quad \text{if} \quad \mu < p \leq \rho \quad \text{and} \quad p \nmid q ,$$

and

$$(29) \qquad \max_{s \in C} \left| \sum_{\rho < p \leq q^{\delta/2}} \frac{\Lambda_{q^{\delta/4}}(p)\chi(p)}{p^s \log p} \right| << \rho^{-\delta} ,$$

where the implicit constant in the $<<$ in (29) depends on σ_1, σ_2, and C. It follows from (27) and (28) that if d is small enough

$$(30) \qquad \max_{s \in C} \left| \sum_{p \leq \mu} \sum_{k=1}^{N} \frac{\chi(p^k)}{p^{ks}} - \sum_{p \leq \mu} \sum_{k=1}^{N} \frac{1}{p^{ks}} \right| \leq \mu^{-\delta}$$

and

$$(31) \qquad \max_{s \in C} \left| \sum_{\mu < p \leq \rho} \frac{\chi(p)}{p^s} - \sum_{\substack{\mu < p \leq \rho \\ p \nmid q}} \frac{e(\theta_p)}{p^s} \right| \leq \mu^{-\delta} .$$

We combine (25), (26), (29), (30), and (31) to obtain

$$(32)\quad \log L(s,\chi) - \sum_{p\le\mu}\sum_{k=1}^{N}\frac{1}{p^{ks}} - \sum_{\substack{\mu<p\le\rho \\ p\nmid q}}\frac{e(\theta_p)}{p^s}$$

$$\ll \mu^{-2\delta} + q^{-\delta^2/8} + \rho^{-\delta} + \mu^{-\delta}$$

where the implied constant depends at most on σ_1, σ_2, and C. Since $\rho > \mu$, if we take q large enough the right-hand side of (32) is

$$\ll \mu^{-\delta} .$$

Define $\ell_\mu(s,q)$ in the statement of the lemma by

$$\ell\mu(s,q) = \sum_{p\le\mu}\sum_{k=1}^{N}\frac{1}{p^{ks}} .$$

This completes the proof. □

§3. Proof of Theorem 5.1

Suppose C and $f(s)$ satisfy the hypotheses of Theorem 5.1. Since C is compact and is contained in the strip $\frac{1}{2} < \sigma < 1$, there are numbers σ_1, σ_2 such that C is in the strip $\frac{1}{2} < \sigma_1 < \sigma < \sigma_2 < 1$. Let $\delta = \sigma_1 - \frac{1}{2}$, let $\ell_\mu(s,q)$ be the entire function of Lemma 5.7, and assume $\mu > 1$. We let $\Lambda_q = \{\log p \mid p \nmid q\}$, where $q \geq 1$. For the counting function $N_{\Lambda_q}(x)$ we have

$$N_{\Lambda_q}(x) = \pi(e^x) + O_q(1) .$$

Since $\pi(x) = \pi(x;\ 1,1)$, we have by Lemma 3.3 for fixed q

$$N_{\Lambda_q}(x) = \mathrm{li}\ e^x + O(e^{x-c\sqrt{x}}) .$$

Obviously

$$N_{\Lambda_q}(x) \ll e^x .$$

Furthermore

$$N_{\Lambda_q}\left(x+\frac{c_1}{x^2}\right) - N_{\Lambda_q}(x) = \int_{e^x}^{e^{x+\frac{c_1}{x^2}}} \frac{dt}{\log t} + O\left(e^{x+c_1/x^2 - e\sqrt{x}}\right)$$

$$\gg \frac{e^{x+\frac{c_1}{x^2}} - e^x}{\log(e^{x+c_1/x^2})} \gg \frac{e^x}{x^3} .$$

Similarly

$$N_{\Lambda_q}(x) - N_{\Lambda_q}\left(x-\frac{c_1}{x^2}\right) \gg \frac{e^x}{x^3} .$$

Thus $N_{\Lambda_q}(x)$ satisfies the hypotheses of Lemma 2.2. Since $f(s) - \ell_\mu(s,q)$ is continuous on C and analytic in the interior of C, there must therefore exist a number $\rho > \mu$ and numbers θ_p, $0 \le \theta_p < 1$, such that

$$(33) \qquad \max_{s\varepsilon C}\left| f(s) - \ell_\mu(s,q) - \sum_{\substack{\mu<p\le\rho \\ p\nmid q}} \frac{e(\theta_p)}{p^s}\right| \ll \mu^{-1/2} \quad ,$$

where the implied constant depends on σ_1, σ_2, C, and Λ_q. On the other hand if q is sufficiently large, we know by Lemma 5.7 that there is a $\chi \pmod q$ such that

$$(34) \quad \max_{s\varepsilon C}\left| \log L(s,\chi) - \ell_\mu(s,q) - \sum_{\substack{\mu<p\le\rho \\ p\nmid q}} \frac{e(\theta_p)}{p^s}\right| \ll \mu^{-\delta} ,$$

the implied constant being dependent on σ_1, σ_2, and C. Combining (33) and (34) yields

$$\max_{s\varepsilon C}\left|\log L(s,\chi) - f(s)\right| \ll \mu^{-1/2} + \mu^{-\delta} \ll \mu^{-\delta} .$$

For large μ we therefore have

$$\begin{aligned} L(s,\chi) &= e^{f(s)} \cdot e^{\log L(s,\chi)-f(s)} \\ &= e^{f(s)}(1 + O(\mu^{-\delta})) \\ &= e^{f(s)} + O((\max_{s\varepsilon C} e^{|f(s)|})\mu^{-\delta}) \\ &= e^{f(s)} + O(\mu^{-\delta}) \end{aligned}$$

uniformly for $s \in C$. Since the error term is independent of q and μ, we conclude on taking μ sufficiently large that for all large q there is a $\chi \pmod q$ such that

$$\max_{s \in C} |L(s,\chi) - e^{f(s)}| < \varepsilon ,$$

where $\varepsilon > 0$ is arbitrary. This proves Theorem 5.1. $\square$

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